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Ken Hendricks

and

Charles Wilson

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C. V. STARR CENTER FOR APPLIED ECONOMICS

NEW YORK UNIVERSITY
FACULTY OF ARTS AND SCIENCE
DEPARTMENT OF ECONOMICS
WASHINGTON SQUARE
NEW YORK, N.Y. 10003

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Ken Hendricks
State University of New York
at Stony Brook

and

Charles Wilson*
New York University

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Abstract

The paper provides a complete characterization of the equilibria for a class of "preemption" games when time is continuous and information is complete. It allows for asymmetric payoffs and an arbitrary time horizon. It extends the analyses of earlier authors to include a class of games in which players move according to a continuous distribution over some interval of the game.

1. Introduction

The returns to a firm from adopting a new technology frequently depend on when it adopts relative to other firms. Most studies focus on markets in which firms have an incentive to adopt preemptively, but where the preferred outcome is to wait and either adopt at some later date or never adopt. The critical issue is the extent to which firms can coordinate their adoption dates and earn some of the profits from delayed adoption. In the models considered by Farrell and Saloner (1986), Fudenberg and Tirole (1985), and Gilbert and Harris (1984), firms generally fail to obtain any of these gains. The incentive for preemption leads to a Bertrand–like outcome in which one firm is certain to adopt as soon as the gains from preemption are positive. In some instances, the models also possess equilibria in which the firms are able to achieve the preferred outcome. Fudenberg and Tirole note that when the gains from preemption are small, there is a continuum of equilibria in which firms adopt jointly at some later date. Farrell and Saloner obtain a similar result, only in their case, the preferred outcome is that neither firm ever adopts.

In this paper we characterize the Nash equilibria for a broad class of these preemption games with complete information. We characterize the conditions under which the Bertrand–like outcomes and the joint adoption outcomes are equilibria. We then establish that, in some instances, there is a continuum of mixed strategy equilibria in which the players wait with probability 1 until some time t, after which they move according to a strictly increasing, continuous distribution function. The possibility of this additional class of equilibria has generally been overlooked in the existing literature.

In the games we analyze, two players must independently choose a time

$t \in [0,1]$ at which to move first. The essential restrictions are that (i) the return to moving first (leading) increases with time and (ii) the return to leading always exceeds the return to following or tying on the **open** interval $(0,1)$. We permit a wide range of assumptions on the return functions at times 0 and 1. In many of the applications of this game, the same model admits several plausible alternative assumptions on the payoffs at these points. Our analysis allows us to examine the sensitivity of equilibrium outcomes as we consider different variations of a model. We address this point in more detail in Section 7.

Many of these models have a common structure which is closely related to a game of timing known as the "noisy" duel.[1] In that game, the return to leading is again increasing with time, but the return to following is strictly decreasing. Karlin (1953) has studied non–zero sum "noisy" duels as a limiting case of the "silent" duel, and provided a characterization of min–max solutions to the game. Pitchik (1982) has recently extended the analysis to Nash equilibria. In each case, these authors find that the equilibria of the "noisy" duel, if any exist, consists of pure strategies. The main difference in the preemption games described above is that the return to following may also be increasing. As we have noted, this may result in a richer class of Nash equilibria.

The paper is organized as follows. In Section 2, we introduce the assumptions which define the preemption game. In Section 3, we derive the properties of the equilibrium strategies on the interior of the strategy space. In Sections 4 and 5, we develop the initial and terminal conditions which equilibrium strategies must satisfy. Section 6 contains a complete characterization of the equilibrium outcomes. In Section 7 we discuss the relation between our results and the adoption models considered in the literature.

2. The Game

We begin with a general description of the game. Two players, a and b, must decide when to make a single move at some time t between 0 and 1.[2] The payoffs are determined as soon as one player moves. In what follows, α refers to an arbitrary player and β to the other player. If player α moves first at some time t, he is called the **leader** and earns a return $L_\alpha(t)$. If player β moves first at time t, then player α is called the **follower** and earns a return $F_\alpha(t)$. If both players move simultaneously at time t, the return to player α is $S_\alpha(t)$. We will refer to $S_\alpha(1)$ as the **terminal** return.

In the strategic form of the game, a pure strategy for player α is a time $t_\alpha \in [0,1]$ at which he plans to move given that neither player moves before that time. Given a strategy pair $(t_a,t_b) \in [0,1]x[0,1]$, the payoff to player α is then defined as follows:

$$P_\alpha(t_a,t_b) = \begin{cases} L_\alpha(t_\alpha) & \text{if} \quad t_\alpha < t_\beta \\ S_\alpha(t_\alpha) & \text{if} \quad t_a = t_b \\ F_\alpha(t_\beta) & \text{if} \quad t_\alpha > t_\beta. \end{cases}$$

2.1 Assumptions on the Payoff Functions

Our first assumption guarantees that the payoff functions are continuous everywhere but on the diagonal.

A1 L_α and F_α are continuous functions on $[0,1]$.[3]

Our next assumption characterizes the class of preemption games we wish to study.[4]

A2 (i) $L_\alpha(t) > F_\alpha(t)$ for $t \in (0,1)$;

(ii) $L_\alpha(t) > S_\alpha(t)$ for $t \in [0,1)$;

(iii) $L_\alpha(t)$ is strictly increasing for $t \in [0,1)$;

Condition (i) requires that the return to leading at any time $t > 0$ strictly exceed the return to following at time t. We do not rule out the possibility that $L_\alpha(0) = F_\alpha(0)$ nor the possibility that $F_\alpha(1) = L_\alpha(1)$. Condition (ii), however, requires that the return to leading strictly exceed the return to tying at <u>all</u> times less than 1. Combined with condition (iii), these conditions imply that, at any time $t < 1$, each player prefers to move before the other but prefers to delay that action as long as possible. Note, however, that, since S_α is not necessarily continuous, our assumptions impose no restrictions on the relation of $S_\alpha(1)$ to either $L_\alpha(1)$ or $F_\alpha(1)$.

As we indicated in the introduction, Assumptions A1 and A2 represent a "reduced form" of many models of the adoption of a new technology. Time 0 corresponds to the first time at which it may be optimal for either firm to adopt. Typically, the underlying model includes an initial interval in which the return to leading is increasing but is less than the return to following. As long as the return to moving simulataneously is less than the return to leading, it is never optimal to move in this interval, and so it can be deleted without loss. Time 1 represents the first instant where the return functions do not satisfy the conditions of Assumption A2. Either the return to leading begins to fall or the returns to following or tying become at least as large as the return to leading. The terminal return, $S_\alpha(1)$, then corresponds to the equilibrium payoff to the continuation game which begins at that point.

In Figure 1, we have illustrated three possible relations between the return functions. As required by Assumption A2, in each case L_α lies above F_α over the open interval (0,1) with L_α strictly increasing throughout. The distinctive features of Figure 1(a) is that at time 0 the return to leading is equal to the return to following, while at time 1 the return to leading is above the return to following. Figure 1(b) illustrates a game in which, initially, the return to leading exceeds the return to following, but the difference converges to 0 as time approaches 1. In this case, $S_\alpha(1)$ may lie between $F_\alpha(0)$ and $L_\alpha(0)$. Figure 1(c) illustrates a game where the return to following is equal to the return to leading at both time 0 and time 1. Although not shown, in each case, S_α can be any function which lies below L_α throughout the half open interval [0,1). Any value of $S_\alpha(1)$ is permissible.

2.2 Equilibrium

It is important for our results to permit agents to randomize across pure strategies. A **mixed strategy** for player α is a probability distribution function G_α on [0,1].[5] If we extend the domain of the payoff functions to the set of all pairs of mixed strategies in the obvious way, then a strategy combination (G_a^*, G_b^*) is an **equilibrium** if $P_\alpha(G_\alpha^*, G_\beta^*) \geq P_\alpha(G_\alpha, G_\beta^*)$ for all mixed strategies G_α, $\alpha = a,b$ and $\beta \neq \alpha$.

For the remainder of the paper, (G_a, G_b) will refer to an equilibrium combination, and $q_\alpha(t)$ will denote the probability with which player α moves at exactly time t. We will repeatedly use the fact that if (G_a,G_b) is a pair of equilibrium distributions, then $P_\alpha(t,G_\beta) = \sup_{v\in[0,1]} P_\alpha(v,G_\beta)$ for any t in the support of G_α.

3. Equilibrium Restrictions on the Interval (0,1)

In this section, we focus on the properties of the equilibrium strategies in the interior of the unit interval.

We begin by establishing that the supports of the strategies of both players must have the same interior (over the interval during which neither player moves with probability 1).

LEMMA 3.1: Suppose $\lim_{t\uparrow t_1} G_\alpha(t) = G_\alpha(t_2) < 1$ for $t_1 < t_2$. Then $G_\beta(t_1-\delta) = \lim_{t\uparrow t_2} G_\beta(t_2)$ for some $\delta > 0$.

PROOF: Suppose $\lim_{t\uparrow t_1} G_\alpha(t) = G_\alpha(t_2) < 1$ for $t_1 < t_2$. Then for any $t \in [t_1,t_2)$, it follows from Assumption A2 that player β prefers to wait until time t_2 to move since there is no chance that player α will move in the intervening interval:

$$P_\beta(t_2,G_\alpha)-P_\beta(t,G_\alpha) = [L_\beta(t_2)-L_\beta(t)][1-G_\alpha(t_2)] > 0.$$

Therefore, if $t_1 = 0$, the lemma is immediate. So suppose $t_1 > 0$. Then, for any $\epsilon > 0$ sufficiently small, there is an arbitrarily small $\delta > 0$ such that $|G_\alpha(t)-G_\alpha(t_2)| < \epsilon$ for any $t \in [t_1-\delta,t_2)$. It then follows from Assumptions A1 and A2 that, for $t \in [t_1-\delta,t_1)$,

$$P_\beta(t_2,G_\alpha)-P_\beta(t,G_\alpha) = [F_\beta(t)-S_\beta(t)]q_\alpha(t) + \int_t^{t_2}[F_\beta(s)-L_\beta(t)]dG_\alpha(s)$$
$$+ [L_\beta(t_2)-L_\beta(t)][1-G_\alpha(t_2)]$$
$$= o(\epsilon)+o(\epsilon)+[L_\beta(t_2)-L_\beta(t)][1-G_\alpha(t_2)]$$
$$> 0.$$

We conclude that player β prefers moving at time t_2 to any time $t \in [t_1-\delta,t_2)$. Q.E.D.

The next Lemma rules out any mass points in the interior of the interval of times which may be reached with positive probability.

LEMMA 3.2: Suppose $t \in (0,1)$. Then $\lim_{v\uparrow t}G_\beta(v) < 1$ implies $q_\alpha(t) = 0$.

PROOF: Suppose, for some $t \in (0,1)$, that $q_\alpha(t) > 0$. Then, for any $\epsilon > 0$, there is an (arbitrarily small) $\delta > 0$ such that (i) $L_\beta(t-\delta)-L_\beta(t+\delta) < \epsilon$, and (ii) $q_\alpha(t-\delta) = 0$ with $G_\alpha(t+\delta)-G_\alpha(t-\delta) < q_\alpha(t)+\epsilon$. It then follows from Assumptions A1 and A2 that, for ϵ and δ chosen sufficiently small,

$$P_\beta(t,G_\alpha)-P_\beta(t-\delta,G_\alpha) = \int_{t-\delta}^{t}[F_\beta(s)-L_\beta(t-\delta)]dG_\alpha(s) + [S_\beta(t)-L_\beta(t-\delta)]q_\alpha(t)$$
$$+ [L_\beta(t)-L_\beta(t-\delta)][1-G_\alpha(t)]$$
$$= o(\epsilon) + [S_\beta(t)-L_\beta(t-\delta)]q_\alpha(t) + o(\epsilon) < 0$$

Similarly, for $v \in (t,t+\delta]$,

$$P_\beta(v,G_\alpha)-P_\beta(t-\delta,G_\alpha) = \int_{t-\delta}^{v}[F_\beta(s)-L_\beta(t-\delta)]dG_\alpha(s)$$

$$+ [S_\beta(v)-L_\beta(t-\delta)]q_\alpha(v) + [L_\beta(v)-L_\beta(t-\delta)][1-G_\alpha(v)]$$
$$= [F_\beta(t)-L_\beta(t-\delta)][q_\alpha(t)+o(\epsilon)] + o(\epsilon) + o(\epsilon) < 0.$$

Consequently, player β will never move in the interval $[t,t+\delta]$. This implies that $\lim_{v\uparrow t} G_\beta(v) = G_\beta(t+\delta)$. Then, if $\lim_{v\uparrow t} G_\beta(v) < 1$, Lemma 3.1 implies that $q_\alpha(t) = 0$, contradicting the hypothesis that $q_\alpha(t) > 0$. Q.E.D.

If G_α is strictly increasing over some interval, then we may use the fact that player α must be indifferent between moving at any two times in the interval to explicitly characterize the equilibrium strategy of player β over this interval in terms of the return functions L_α and F_α.

For $0 \le t_0 < t \le 1$, define

$$I_\beta(t_0,t) = \exp[\textstyle\int_{t_0}^{t}[dL_\alpha(v)/(F_\alpha(v)-L_\alpha(v))]].$$ [6]

LEMMA 3.3: Suppose G_α is strictly increasing over the interval $[t_0,t_1]$. Then, for $t_0 > 0$ and $t \in (t_0,t_1)$, $G_\beta(t) < 1$ implies

(3.1) $G_\beta(t) = 1 - [1-G_\beta(t_0)]I_\beta(t_0,t).$

PROOF: Suppose that G_α is strictly increasing over the interval $[t_0,t_1]$. Then, since $G_\alpha(t) < 1$ for $t < t_1$, it follows from Lemma 3.2 that G_β is continuous on $(0,t)$. Therefore, for any $t \in [t_0,t_1)$,

(3.2) $0 = P_\alpha(t,G_\beta) - P_\alpha(t_0,G_\beta)$

$$= \textstyle\int_{t_0}^{t}[F_\alpha(v)-L_\alpha(t_0)]dG_\beta(v) + [1-G_\beta(t)][L_\alpha(t)-L_\alpha(t_0)]$$

Since G_β and L_α are both monotonic and L_α is continuous, we may apply the formula for integration by parts (Rudin (1964), p.122) to obtain

$$(3.3)\quad [1-G_\beta(t)][L_\alpha(t)-L_\alpha(t_0)] = \int_{t_0}^{t}[1-G_\beta(v)]dL_\alpha(v) - \int_{t_0}^{t}[L_\alpha(v)-L_\alpha(t_0)]dG_\beta(v).$$

Substituting (3.3) into (3.2) and rearranging terms then yields, for all $t \in [t_0,t_1)$,

$$(3.4)\quad \int_{t_0}^{t} [L_\alpha(v)-F_\alpha(v)][1-G_\beta(v)] \left[\frac{dG_\beta(v)}{1-G_\beta(v)} - \frac{dL_\alpha(v)}{L_\alpha(v)-F_\alpha(v)} \right] = 0.$$

But, since $[L_\alpha(v)-F_\alpha(v)][1-G_\beta(v)] > 0$ for all $v \in [t_0,t]$, equation (3.4) implies that

$$\int_{t_0}^{t} dG_\beta(v)/[1-G_\beta(v)] = -\int_{t_0}^{t} dL_\alpha(v)/[F_\alpha(v)-L_\alpha(v)].$$

Employing a change of variable (Rudin (1964), p.122–124), we may then apply the fundamental theorem of calculus to obtain:

$$\log[1-G_\beta(t)] = \log[1-G_\beta(t_0)] + \int_{t_0}^{t} dL_\alpha(v)/[F_\alpha(v)-L_\alpha(v)].$$

Taking antilogs and rearranging terms then yields equation (3.1). Q.E.D.

Note that, if L_α is continuously differentiable, then equation (3.1) is simply the solution to the differential equation

$$G'_\beta(t)/[1-G_\beta(t)] = L'_\alpha(t)/[L_\alpha(t)-F_\alpha(t)]$$

with initial condition $G_\beta(t_0) \in [0,1]$. In this case, G_β has a continuous density function g_β over the interval (t_0,t_1).

We establish next that if neither player moves with probability 1 at time 0, then the game cannot end with certainty until time 1.

LEMMA 3.4: Suppose $G_\alpha(0) < 1$. Then $G_\beta(0) < 1$ implies $G_\beta(t) < 1$ for $t < 1$.

PROOF: Let $\hat{t} = \sup\{t \geq 0\colon G_\alpha(t) < 1 \text{ for } \alpha = a,b\}$ be the earliest time by which one of the players has moved with certainty. The lemma is equivalent to the requirement that $\hat{t} \notin (0,1)$. Suppose $0 < \hat{t} < 1$.

We will show first that the strategy of one of the players must have a mass point at $\hat{t}$. Suppose not. Then, for some player β, G_β is strictly increasing over an interval $(t',\hat{t})$ and $\lim_{t\uparrow\hat{t}} G_\beta(t) = 1$. Then Lemma 3.3 combined with Assumption A2 implies G_α is strictly increasing over the interval $(t',\hat{t})$. Applying Lemma 3.3 again, it then follows from Assumption A2 that $\lim_{t\uparrow\hat{t}} G_\beta(t) = 1 - [1-G_\beta(t')]I_\beta(t',\hat{t}) < 1$. A contradiction.

But if $q_\beta(\hat{t}) > 0$, then Lemma 3.2 implies that $\lim_{t\uparrow\hat{t}} G_\alpha(t) = 1$. The definition of $\hat{t}$ then implies that G_α is strictly increasing over some interval $(t',\hat{t})$. It then follows from Assumption A2 and Lemma 3.3 that $\lim_{t\uparrow\hat{t}} G_\alpha(t) < 1$. This contradiction proves the lemma. Q.E.D.

Define

$$t^* = \inf\{t \le 1: G_\alpha \text{ is strictly increasing on } (t,1] \text{ for } \alpha = a,b\} \cup \{1\}.$$

Given any strategy combination in which neither player moves with probability 1 before time 1, t^* is the end of the last interval during which one of the players moves with probability 0. Otherwise, it is set equal to 1. Combining Lemmata 3.1 to 3.4, we can show that, unless one of the players moves with probability 1 at time 0, neither player ever moves in the interval $(0,t^*)$.

LEMMA 3.5: Suppose $G_\alpha(0) < 1$. Then

(i) $G_\beta(t) = q_\beta(0)$ for $0 \le t < t^*$, and

(ii) $G_\beta(t) = 1 - [1-q_\beta(0)]I_\beta(t^*,t)$ for $t^* \le t < 1$.

PROOF: Suppose that $q_\alpha(0) < 1$. Then, given $t^* < 1$, Lemma 3.3 and the right-continuity of G_β imply that $G_\beta(t) = 1 - [1-G_\beta(t^*)]I_\beta(t^*,t)$ for $t^* \le t < 1$. Furthermore, if $t^* \in (0,1)$, then Lemmata 3.2 and 3.4 imply that $q_\beta(t^*) = 0$. Therefore, the lemma will be proved if we can establish part (i). Note that it is trivially true if $q_\beta(0) = 1$. Therefore, suppose $q_\beta(0) < 1$. Let

$$t' = \inf\{v \ge 0: G_\beta(v) = \lim_{t\uparrow t^*} G_\beta(t)\}.$$

We need to show that $t' = 0$ when $t^* > 0$.

Suppose first that $t' = t^* > 0$. Then the definition of t^* implies that, for some $t'' < t'$, $G_\alpha(t'') = \lim_{t\uparrow t^*} G_\alpha(t)$. Lemma 3.4 then implies that $\lim_{t\uparrow t^*} G_\alpha(t) < 1$. Combined with Lemma 3.1, this implies that $G_\beta(t'') = \lim_{t\uparrow t^*} G_\beta(t)$, contradicting the definition of t'.

Suppose next that $0 < t' < t^*$. Then, since Lemma 3.4 implies that $G_\alpha(t') < 1$, it follows from Lemma 3.2 that $\lim_{t\uparrow t'} G_\beta(t) = \lim_{t\uparrow t^*} G_\beta(t^*) < 1$. But then Lemma 3.1 implies that, for some $\delta > 0$, $G_\alpha(t'-\delta) = \lim_{t\uparrow t^*} G_\alpha(t)$. Applying Lemma 3.1 again then yields $G_\beta(t'-\delta) = \lim_{t\uparrow t^*} G_\beta(t)$, contradicting the definition of t'. Q.E.D.

Lemma 3.5 implies that the support of the equilibrium strategies when neither player moves with probability 1 at time 0 is composed of at most $\{0\}$ and an interval $[t^*,1]$. Furthermore, any differences among the equilibrium strategies of player β must occur in the values of either $q_\beta(0)$ or t^*.

4. Equilibrium Restrictions at Time 0

In this Section, we consider the equilibrium restrictions on the strategies at time 0. The possibility for mass points in the equilibrium strategies at time 0 depends on the relation between the return to moving simultaneously and the return to following at time 0.

We consider first the conditions that are necessary for the existence of an equilibrium in which one of the players moves immediately. Define

$$I_\beta(0,0) = \lim_{t\downarrow 0} I_\beta(0,t)].$$

LEMMA 4.1: Suppose $G_\alpha(0) = 1$. Then $I_\beta(0,0) = 1$, and

(i) $q_\beta(0) > 0$ implies $S_\alpha(0) \geq F_\alpha(0)$ and $S_\beta(0) \geq F_\beta(0)$.

(ii) $q_\beta(0) < 1$ implies $S_\beta(0) \leq F_\beta(0)$;

PROOF: Suppose $G_\alpha(0) = 1$.

If $q_\beta(0) > 0$ is an optimal response, then player β must not earn a higher return by waiting an instant:

$$0 \geq \lim_{t \downarrow 0} P_\beta(t,G_\alpha) - P_\beta(0,G_\alpha) = F_\beta(0) - S_\beta(0).$$

Similarly, $q_\beta(0) > 0$ implies that $q_\alpha(0) = 1$ is an optimal response only if

$$0 \geq \lim_{t \downarrow 0} P_\alpha(t,G_\beta) - P_\alpha(0,G_\beta) = q_\beta(0)[F_\alpha(0) - S_\alpha(0)].$$

This establishes (i).

If $q_\beta(0) < 1$, then player β receives the same return from moving at some $t > 0$ as from moving immediately:

$$0 \leq P_\beta(t,G_\alpha) - P_\beta(0,G_\alpha) = F_\beta(0) - S_\beta(0).$$

This establishes (ii).

All that remains is to show that $q_\alpha(0) = 1$ implies $I_\beta(0,0) = 1$. Note first that, whenever $F_\alpha(0) \leq S_\alpha(0)$, Assumptions A1 and A2 imply that $F_\beta(t)$ is bounded away from $L_\beta(t)$ for small t. This implies that $I_\alpha(0,0) = 1$. But if $S_\alpha(0) < F_\alpha(0)$, then condition (i) implies that $q_\beta(0) = 0$. In this case, therefore, we must be able to construct a mixed strategy for player β which confers no gain to player α from waiting.

Suppose that $q_\beta(0) = 0$ and $P_\alpha(t,G_\beta) \leq P_\alpha(0,G_\beta)$ for $t > 0$. Then, for any $\epsilon > 0$, there is a $t \in (0,\epsilon)$ such that

But if $I_\beta(0,0) < 1$, then, $\int_0^u dL_\alpha/[L_\alpha(v)-F_\alpha(v)] = \infty$ which implies that $\lim_{u\downarrow 0} G_\beta(u) = \infty$. Q.E.D.

The implications of Lemma 4.1 are summarized in Table 1. Typical elements represent $(q_a(0),q_b(0))$. The value of x is any number in the interval $[0,1)$.

TABLE 1. Possible Equilibrium Outcomes Where One Player Moves Immediately

	$S_b(0) < F_b(0)$	$S_b(0) = F_b(0)$	$S_b(0) > F_b(0)$
$S_a(0) < F_a(0)$	$(1,0)^b,(0,1)^a$	$(0,1),(1,0)^b$	$(0,1)$
$S_a(0) = F_a(0)$	$(1,0),(0,1)^a$	$(1,x),(x,1)$	$(x,1)$
$S_a(0) > F_a(0)$	$(1,0)$	$(1,x)$	$(1,1)$

[a] If $I_a(0,0) = 1$.

[b] If $I_b(0,0) = 1$.

The integral condition at time 0, $I_\beta(0,0) = 1$, must be satisfied in order to define a mixed strategy for player β which makes player α indifferent to moving at any time near 0. Since $I_\beta(0,0) = 1$ if $F_\alpha(0) < L_\alpha(0)$, Assumption A2 implies that it is always satisfied if $F_\alpha(0) \le S_\alpha(0)$. Thus, the integral condition imposes an additional restriction on the payoffs at time 0 only when $S_\alpha(0) < F_\alpha(0)$.

We consider next the equilibrium restrictions at time 0 for strategy combinations in which there is a positive probability of the game not ending at time 0. There are two cases to consider as determined by the value of t^*.

LEMMA 4.2: Suppose $t^* = 0$. Then

(i) $I_a(0,0) = I_b(0,0) = 1$.[7]

(ii) $S_\alpha(0) > F_\alpha(0)$ implies $q_\beta(0) = 0$.

(iii) $S_\alpha(0) < F_\alpha(0)$ implies $q_a(0)q_b(0) = 0$.

PROOF: Condition (i) follows from Lemma 3.3 and the requirement that G_α be right-continuous. To establish conditions (ii) and (iii), note that, if $t^* = 0$, the payoffs earned by each player from waiting an instant must be no less than their payoffs from moving immediately:

$$(4.1)\quad 0 \le \lim_{t\downarrow 0} P_\alpha(t,G_\beta) - P_\alpha(0,G_\beta) = q_\beta(0)[F_\alpha(0) - S_\alpha(0)].$$

If $S_\alpha(0) > F_\alpha(0)$, relation (4.1) implies $q_\beta(0) = 0$. On the other hand, if $S_\alpha(0) < F_\alpha(0)$ and $q_\alpha(0) > 0$, then (4.1) must hold with equality, which again implies that $q_\beta(0) = 0$. Q.E.D.

The implications of Lemma 4.2 are summarized in Table 2. Typical elements represent $(q_a(0),q_b(0))$. The value of x is any number in the interval [0,1). If $S_\alpha(0) < F_\alpha(0)$, then it is assumed that $I_\beta(0,0) = 1$.

TABLE 2. Possible Initial Mass Points When $t^*=0$

	$S_b(0) < F_b(0)$	$S_b(0) = F_b(0)$	$S_b(0) > F_b(0)$
$S_a(0) < F_a(0)$	(x,0),(0,x)	(0,x),(x,0)	(0,x)
$S_a(0) = F_a(0)$	(x,0),(0,x)	(x,x),(x,x)	(0,x)
$S_a(0) > F_a(0)$	(x,0)	(x,0)	(0,0)

LEMMA 4.3: Suppose $t^* \in (0,1)$. Then either

(i) $q_a(0) = q_b(0) = 0$, or

(ii) $S_\beta(0) > F_\beta(0)$ and $q_\beta(0) = [L_\alpha(t^*)-L_\alpha(0)]/[L_\alpha(t^*)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$ for $\beta = a,b$.

PROOF: Suppose $t^* \in (0,1)$. If $q_\beta(0) = 0$, then it follows from Lemma 3.5 that player β never moves in the interval $[0,t]$. Therefore, since L_α is strictly increasing, player α receives a higher payoff from moving at time t^* than at time 0:

$$P_\alpha(t^*,G_\beta)-P_\alpha(0,G_\beta) = L_\alpha(t^*)-L_\alpha(0) > 0.$$

Therefore, $q_\alpha(0) = 0$. On the other hand, if $q_\alpha(0) > 0$, then player α must be indifferent between moving at time 0 and time t^*:

$$(4.2)\quad 0 = P_\alpha(t^*,G_\beta)-P_\alpha(0,G_\beta) = q_\beta(0)[F_\alpha(0)-S_\alpha(0)] + [1-q_\beta(0)][L_\alpha(t^*)-L_\alpha(0)].$$

This implies condition (ii). Q.E.D.

5. Equilibrium Restrictions at Time 1

In this section, we consider the restrictions on the strategies at time 1. Once again there are two cases to consider as determined by the value of t^*.

Define

$I_\alpha(1,1) \equiv \lim_{t\uparrow 1} I_\alpha(t,1)$.

LEMMA 5.1: Suppose $t^* < 1$ and $I_\beta(1,1) = 1$.[8] Then

(i) $L_\alpha(1) \geq S_\alpha(1)$;

(ii) $I_\alpha(1,1) > 0$ implies $L_\alpha(1) = S_\alpha(1)$.

PROOF: Suppose $t^* < 1$. Thus, $q_\alpha(0) < 1$ for α =a,b. Then, since G_α is strictly increasing over $(t^*,1)$, the payoff to player α from moving just before time 1 must be at least as large as his payoff from waiting until time 1:

$$(5.1)\quad 0 \geq P_\alpha(1,G_\beta) - \lim_{t\uparrow 1} P_\alpha(t,G_\beta) = q_\beta(1)[S_\alpha(1)-L_\alpha(1)].$$

If $I_\beta(1,1) > 0$, then Assumption A2 implies that $I_\beta(t,1) > 0$ for all $t \in (0,1)$. It then follows from Lemma 3.5 that $q_\beta(1) = [1-q_\beta(0)]I_\beta(t^*,1) > 0$. Part (i) then follows from equation (5.1).

To establish part (ii), note that, if $I_\alpha(1,1) > 0$, then $q_\alpha(1) > 0$, which implies relation (5.1) must hold with equality. This establishes (ii). Q.E.D.

LEMMA 5.2: Suppose $t^* = 1$ and $q_a(0)q_b(0) < 1$. Then

(i) $S_\alpha(1) \geq L_\alpha(1)$, α = a,b, and

(ii) $q_\alpha(1) < 1$ implies $q_\beta(1) = [S_\alpha(0)-F_\alpha(0)]/[S_\alpha(1)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$.

PROOF: Suppose that $t^* = 1$ and $q_\alpha(0) < 1$ for α = a,b. Then it follows from the Lemma 3.5 that, for α = a,b, $q_\alpha(1) = 1-q_\alpha(0) > 0$. Therefore, the payoff to either player α from moving at time 1 must be at least as great as his payoff from moving an instant earlier:

$$0 \geq P_\alpha(1,G_\beta) - \lim_{t\uparrow 1} P_\alpha(t,G_\beta) = q_\beta(1)[S_\alpha(1) - L_\alpha(1)].$$

This establishes (i). Suppose further that $q_\alpha(1) < 1$. Then player α must be indifferent between moving at times 0 and 1:

$$0 = P_\alpha(1,G_\beta) - P_\alpha(0,G_\beta) = [1-q_\beta(1)][F_\alpha(0)-S_\alpha(0)] + q_\beta(1)[S_\alpha(1)-L_\alpha(0)].$$

Rearranging terms yields part (ii). Q.E.D.

Note that Part (ii) of Lemma 5.2 implies that $S_\alpha(0) > L_\alpha(0)$ for both players α whenever $t^* = 1$ and both $q_a(1)$ and $q_b(1)$ are less than 1.

6. A Complete Characterization of the Equilibria

Using the restrictions derived in Sections 3 to 5, we may completely characterize the set of equilibria. First, we characterize the "degenerate" equilibria in which either one of the players moves at time 0 or both wait until time 1. Next, we characterize the "nondegenerate" equilibria in which the strategies of both players are increasing over some interval of the game. We conclude with a statement of the conditions under which at least one equilibrium exists.

6.1 Degenerate Equilibria

To facilitate the statement of our results, we distinguish between two types of degenerate equilibria. Our first theorem characterizes the set of equilibria in which one of the players moves with probability 1 at time 0.

THEOREM 6.1: The following conditions characterize the set of equilibrium outcomes for which, for some player β, $q_\beta(0) = 1$:

(i) $S_\alpha(0) < F_\alpha(0)$, $I_\alpha(0,0) = 1$, and $q_\alpha(0) = 0$;

(ii) $S_\alpha(0) = F_\alpha(0)$, $S_\beta(0) \geq F_\alpha(0)$, and $q_\alpha(0) \in [0,1]$;

(iii) $S_\alpha(0) > F_\alpha(0)$, $S_\beta(0) > F_\alpha(0)$, and $q_\alpha(0) = 1$.

PROOF: The necessity of conditions (i) to (iii) follows from Lemma 4.1. To establish the sufficiency of these conditions, we need to construct a strategy for player α such that player β has no incentive not to move with probability 1 at time 0. Suppose the strategy of player α is defined as

$$G_\alpha(t) = q_\alpha(0)+[[1-q_\alpha(0)][1-I_\alpha(0,t]/[1-I_\alpha(0,1)].$$

Note that each of the conditions (i) to (iii) imply that $I_\alpha(0,0) = 1$. Therefore, G_α is a continuous, strictly increasing function on $[0,1]$ such that $G_\alpha(0) = q_\alpha(0)$ and $\lim_{t\uparrow 1} G_\alpha(t) = 1$. Then, for $t \in [0,1]$,

$$\begin{aligned}
&P_\beta(t,G_\alpha)-P_\beta(0,G_\alpha) \\
&= q_\alpha(0)[F_\beta(0)-L_\beta(0)] + \textstyle\int_0^t[F_\beta(v)-L_\beta(0)]dG_\alpha(v) + [1-G_\alpha(t)][L_\beta(t)-L_\beta(0)] \\
&= q_\alpha(0)[F_\beta(0)-L_\beta(0)] - [[1-q_\alpha(0)]/[1-I_\alpha(0,1)]] \\
&\quad [\textstyle\int_0^t[F_\beta(v)-L_\beta(0)]dI_\alpha(0,v) - [L_\beta(t)-L_\beta(0)][I_\alpha(0,t)-I_\alpha(0,1)]] \\
&\leq q_\alpha(0)[F_\beta(0)-L_\beta(0)] - [[1-q_\alpha(0)]/[1-I_\alpha(0,1)]] \\
&\quad [\textstyle\int_0^t[F_\beta(v)-L_\beta(0)]dI_\alpha(0,v) - I_\alpha(0,t)[L_\beta(t)-L_\beta(0)]]
\end{aligned}$$

(integrating by parts)

$= q_\alpha(0)[F_\beta(0)-L_\beta(0)] - [[1-q_\alpha(0)]/[1-I_\alpha(0,1)]]$

$[\int_0^t[[F_\beta(v)-L_\beta(v)]dI_\alpha(0,v) - I_\alpha(0,v)dL_\beta(v)]$

(by the definition of I_α)

$= q_\alpha(0)[F_\beta(0)-L_\beta(0)] \le 0.$

Therefore, $q_\beta(0) = 1$ is a best response to G_α. It is easy to verify that, under the conditions (i) to (iii), G_α is also a best response to G_β. Q.E.D.

Our next theorem defines the set of equilibria in which both players move with positive probability at time 1, but neither player ever moves in the interval (0,1).

THEOREM 6.2: The following conditions characterize the equilibria for which $q_\beta(1) = 1-q_\beta(0) > 0$ for $\beta = a,b$.

(i) $S_\alpha(1) \ge L_\alpha(1)$ for $\alpha = a,b$, and $q_a(1) = q_b(1) = 1$;

(ii) $S_\alpha(1) \ge L_\alpha(1)$, $S_\alpha(0) > F_\alpha(0)$, and

$q_\beta(1) = [S_\alpha(0)-F_\alpha(0)]/[S_\alpha(1)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$ for $\alpha = a,b$.

PROOF: The necessity of these conditions follow from Lemma 5.2. Sufficiency is established by inspection. Q.E.D.

If we ignore the nongeneric cases in which either $S_\alpha(0) = F_\alpha(0)$ or $S_\alpha(1) = L_\alpha(0)$, the characterization of the degenerate equilibria may be summarized reasonably succinctly. If both players earn a higher return at time 1 than leading an instant before, then there is always an equilibrium in which both players wait until time 1 with probability 1. In addition, if, at time 0, both players prefer moving simultaneously to following, there is a unique equilibrium in which both players move at both times 0 and 1 with positive probability. Otherwise, the only degenerate equilibrium is for at least one player to move at time 0 with probability 1.

The possible degenerate equilibria of this last type also depend on the properties of the return functions near time 0. If both players prefer to move simultaneously rather than follow, then there is always a equilibrium in which both players move immediately with probability 1. Otherwise, one of the players must wait with probability 1. For such an equilibrium to exist, however, it must be possible to construct a strategy which puts enough weight on moving just after time 0 to induce the other player to move immediately. This requires that the integral condition, $I_\beta(0,0) = 1$, be satisfied.

The generic degenerate equilibria are summarized in Table 3. Typical elements represent $(q_a(0),q_b(0))$.

TABLE 3. An Index of the Degenerate Equilibrium Strategies

	$S_b(0) < F_b(0)$	$S_b(0) > F_b(0)$
$S_a(0) < F_a(0)$	$(0,0)^a$ $(1,0)^d$, $(0,1)^c$	$(0,0)^a$ $(0,1)^c$
$S_a(0) > F_a(0)$	$(0,0)^a$ $(1,0)^d$	$(0,0)^a$ $(y_a,y_b)^b$ $(1,1)$

[a] If $S_\alpha(1) > L_\alpha(1)$ for $\alpha = a,b$.

[b] $y_\beta = [S_\alpha(0)-F_\alpha(0)]/[S_\alpha(1)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$ for $\beta = a,b$.

[c] If $I_a(0,0) = 1$.

[d] If $I_b(0,0) = 1$.

6.2 Nondegenerate Equilibria

All that remains is to characterize the equilibria in which at least one of the players moves with positive probability in the interval $(0,1)$.

THEOREM 6.3: (a) There is an equilibrium such that $q_\alpha(0)+q_\alpha(1) < 1$ for some α only if one of the following conditions is satisfied:

(i) $I_a(1,1) = I_b(1,1) = 0$;

(ii) $I_\alpha(1,1) = 0$ and $L_\alpha(1) \geq S_\alpha(1)$ for some α; or

(iii) $L_\alpha(1) = S_\alpha(1)$ for $\alpha = a,b$.

(b) If one the conditions of part (a) are satisfied, then, in addition to the equilibria defined in Theorems 6.1 and 6.2, the strategy combinations defined, for $\beta = a,b$, by

$$G_\beta(t) = 1 - [1-q_\beta(0)]I_\beta(t_0,t) \quad \text{for} \quad t \in [t_0,1)$$

characterize the set of equilibria under the following restrictions on t_0 and $(q_a(0),q_b(0)) \in [0,1)\times[0,1)$:

(i) $t_0 \in (0,1)$ and one of the following conditions are satisfied:

(a) $q_a(0) = q_b(0) = 0$;

(b) $S_\beta(0) > F_\beta(0)$ and

$q_\beta(0) = [L_\alpha(t_0)-L_\alpha(0)]/[L_\alpha(t_0)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$ for $\beta = a,b$;

(ii) $t_0 = 0$ and one of the following conditions are satisfied:

(a) $S_\alpha(0) \leq F_\alpha(0)$ and $I_\alpha(0,0) = 1$ for $\alpha = a,b$, and

$q_a(0)q_b(0) = 0$;

(b) $S_\alpha(0) = F_\alpha(0)$, $S_\beta(0) > F_\beta(0)$, and $q_\alpha(0) = 0$ for some α;

(c) $S_\alpha(0) = F_\alpha(0)$ for $\alpha = a,b$;

PROOF: The necessity of the conditions in part (a) follows from Lemma 5.1. The necessity of conditions in part (b) follows from Lemmata 4.2 and 4.3. The theorem then follows upon verifying that each of these strategy pairs are best responses. Q.E.D.

Any nondegenerate pair of strategies must satisfy Lemma 3.5. Igoring the cases in which either $S_\alpha(0) = F_\alpha(0)$ or $S_\alpha(1) = L_\alpha(1)$, nondegenerate equilibria exist only if (i) these strategies imply that both players move with probability 1 before time 1, or (ii) one player moves with probabilty 1 before time 1 and prefers his return to leading just before time 1 to his return at time 1. A necessary condition for player α to move before time 1 with certainty is that his return to following converge to his return to leading as time approaches 1.

As long as the terminal conditions at time 1 are satisfied, there is a one parameter family of nondegenerate equilibria, indexed by the value of t_0 (possibly representing two distinct equilibria). Furthermore, unless both players prefer moving simultaneously to following at time 0, there is also a one parameter family of equilibria in which the strategies of both players are strictly increasing throughout the interval [0,1]. These equilibria are indexed by the probability with which one of the players moves at time 0.

The class of nondegenerate equilibria are summarized in Table 4. As before, typical elements represent $(q_a(0),q_b(0))$. When $t_0 = 0$, the table corresponds to the case where $I_\alpha(0,0) = 1$ for $\alpha = a,b$.

TABLE 4. An Index of the Nondegenerate Equilibrium Strategies

	$S_b(0) < F_b(0)$	$S_b(0) > F_b(0)$
$S_a(0) < F_a(0)$	$t_0 = 0$: $(x,0),(0,x)$ $t_0 > 0$: $(0,0)$	$t_0 = 0$: $(0,x)$ $t_0 > 0$: $(0,0)$
$S_a(0) > F_a(0)$	$t_0 = 0$: $(x,0)$ $t_0 > 0$: $(0,0)$	$t_0 = 0$: $(0,0)$ $t_0 > 0$: $(0,0),(y_a,y_b)$[a]

[a] $q_\beta(0) = [L_\alpha(t_0)-L_\alpha(0)]/[L_\alpha(t_0)-L_\alpha(0)+S_\alpha(0)-F_\alpha(0)]$ for $\beta = a,b$.

6.3 The Existence of Equilibrium

Theorems 6.1 to 6.3 can be combined to yield the following existence result.

COROLLARY 6.1: An equilibrium exists if and only if one of the following conditions are satisfied:

(i) $I_\alpha(0,0) = 1$ for some player α;

(ii) $S_\alpha(1) \geq L_\alpha(1)$ for $\alpha = a,b$; or

(iii) the conditions of theorem 6.3 (a) are satisfied.

Condition (i) of Corollary 6.1 guarantees the existence of a degenerate equilibrium in which one of the players moves at time 0 with probability 1. Condition (ii) guarantees the existence of an equilibrium in which both wait until time 0 with probability 1. Condition (iii) guarantees the existence of a nondegenerate equilibrium.

One important case for which there may be no equilibrium is illustrated in Figure 1(a). Both the terminal return and the return from following are less than the return from leading at time 1, the return from following and leading are equal at time 0, and the derivative of the return to leading is positive at time 0. Under these assumptions, it can be shown that $I_\alpha(0,0) = 0$ and $I_\alpha(1,1) = 1$. It then follows from Lemma 3.5 that in any nondegenerate equilibrium both players must wait until time 1 with positive probability. But if $S_\alpha(1) < L_\alpha(1)$, this cannot be optimal. Consequently, one of the players must move immediately in equilibrium. Furthermore, since both players prefer following at time 0 to moving simultaneously, only one player, say player β, can move immediately. To enforce this behavior, however, it must be possible to define a strategy for player α with enough probability concentrated near time 0 to induce player β to move immediately which is not possible since $I_\alpha(0,0) = 0$.

This problem does not arise if the game is formulated in discrete time. In this case, there is an equilibrium in which one of the players moves with certainty in either the first or second period of the game (depending on whether F_α is increasing or decreasing at 0) and the other player moves with sufficiently high probability in the following periods to keep the player from waiting. None of these equilibria correspond to continuous time equilibria, however, because the strategies converge to a mass point at time 0 as the discreteness of time is made ever finer. Essentially, the problem is that there are no "second" or "third" periods in continuous time. (See Hendricks and Wilson (1985) for a more detailed discussion of this point.)

Gilbert and Harris (1984) resolve the existence problem by essentially redefining the payoffs when two players move simultaneously. They are determined as if one of the players, say player b, can observe the action of player a at any time t before he commits himself to move at that time. In the framework of this paper, the effect is to make the return to tying equal to the return to leading and following at time 0. Consequently, moving immediately is a best response for both players. Fudenberg and Tirole (1985) use another approach to solve the existence problem. They enlarge the strategy space to include "intensity" functions for which the payoffs are defined as if the players coordinated their moves so that only one player ever moves first. This enlarged game admits a symmetric equilibrium in which each player earns the leading (and following) return at time 0.[9]

7. Applications

We conclude with a brief discussion of the relation between our analysis and the models of the adoption of a new technology mentioned in the introduction. Return to Figure 1. Each of the three cases corresponds to a different variation of the adoption game.

Figure 1(a) illlustrates the return functions in the Fudenberg–Tirole model when the payoffs are symmetric. After an initial interval where the return to following exceeds the return to leading, the game begins at time 0 where the return to adopting a new technology first equals the return to following. Time 1 corresponds to the date at which the returns to leading begin to fall. They suppose that the return to following is strictly less than the return to leading at this point. However, there are two possibilities for the value of $S_\alpha(1)$. Upon reaching period 1, the equilibrium is either for one of the players to move immediately and the other to follow or for both players to wait until some later date when the return to joint adoption is even larger. In the first case, $S_\alpha(1) = L_\alpha(1)$ for the leader but $S_\beta(1) = F_\beta(1) < L_\alpha(1)$ for the follower. In the second case, $S_\alpha(1) \geq L_\alpha(1)$ for both players α.

In the Section 6.3, we argued that because $I_\alpha(0,0) = 1$ in this example, Theorem 6.1 implies that it can never be an equilibrium for either one of the players to move immediately. Furthermore, since $I_\alpha(1,1) = 0$, Theorems 6.2 and 6.3(a) also imply that there is no mixed strategy equilibria except in the special case where $S_\alpha(1) = L_\alpha(1)$ for both firms. In general, the only possibility for equilibrium is for both players to wait until time 1. It exists only if the returns from delayed joint adoption are no less than the return to leading at time 1.

The reason why the returns to leading in the Fudenberg–Tirole model exceed the returns to following at time 1 is because the optimal response of the follower to adoption at time 1 is to wait until some time later before adopting. Hence, the gains to preemption at time 1 are positive. However, it is equally plausible to suppose that, at some point, the optimal response of a follower is to adopt an instant later while the returns to leading are still increasing. In this case, the return to following is equal to the return to leading at time 1. This is the case illustrated in 1(c). The returns to leading are increasing throughout, but at time 1, $S_\alpha(1) = L_\alpha(1) = F_\alpha(1)$. If $dL_\alpha(1)/dt > 0$, then $I_\alpha(1,1) = 0$. In this case, it follows from Theorems 6.2 and 6.3 that there is a continuum of nondegenerate equilibria in this model as well as a degenerate equilibrium in which both players wait until time 1.

Figure 1(b) pictures the return functions for a variation on a model studied by Farell and Saloner.[10] In their model, the adoption of the new technology leads to a **decrease** in the flow of profits to both firms. However, the firm which adopts first suffers a smaller decrease than the follower. These assumptions imply that the returns to leading and following are an increasing function of the date of the first adoption with the return to leading at any t exceeding the returns to following. Consequently, even though the firms earn a lower profit by adopting the new technology, each firm has an incentive to preempt in order to avoid the losses from being preempted. As the date of preemption gets large, the returns to leading and following converge to the returns associated with the outcome in which neither firm ever adopts.

After normalizing time to fit our framework,[11] it follows immediately that $F_\alpha(1) = L_\alpha(1) = S_\alpha(1)$. Furthermore, it can be shown that $\lim_{t\to 1}(L_\alpha'(t)/[F_\alpha(t)-L_\alpha(t)] > 0$. Consequently, the integral condition, $I_\alpha(1,1) = 0$, is satisfied. It then follows from Theorems 6.2 and 6.3 that there is a degenerate equilibrium in which neither firm ever adopts and a continuum of nondegenerate equilibria. The properties of the equilibria in which at least one of the firms adopts immediately depends on the relation between $S_\alpha(0)$ and $F_\alpha(0)$. If the return to following at time 0 exceeds the return to joint adoption, then only one firm may move immediately. If the return to following lies below the return to joint adoption, then, if one firm moves immediately with positive probability, the other firm must move with positive probability as well.

FIGURE 1. Three Possible Patterns for the Return Functions

Figure 1(a)

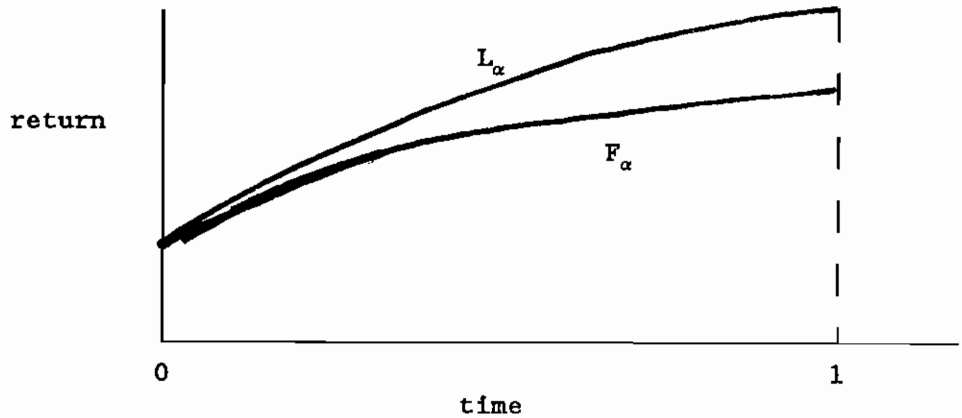


Figure 1(b)

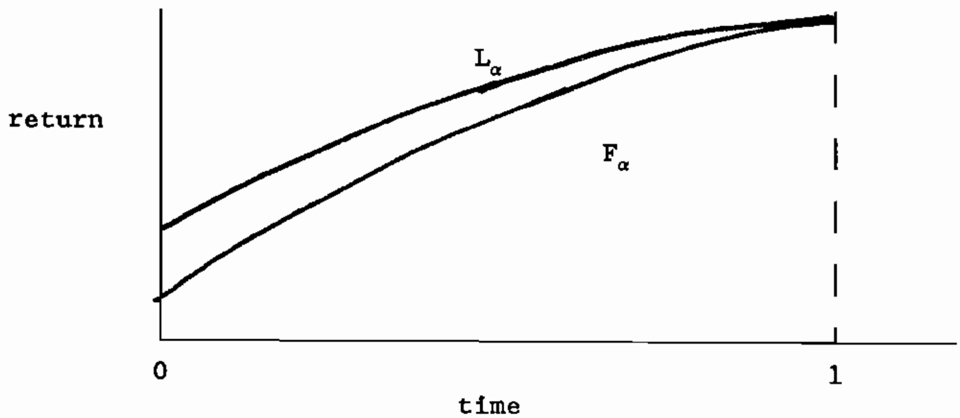
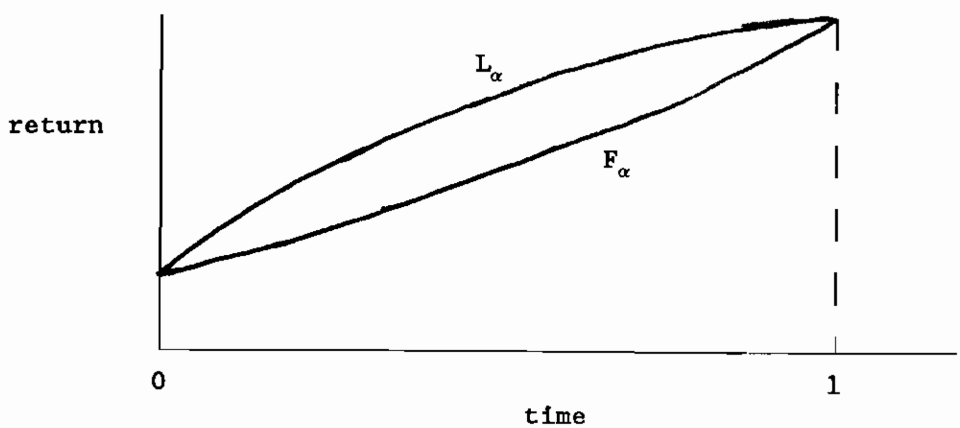


Figure 1(c)



Footnotes

[1] The games are called "noisy" because the payoff to the follower depends only on when the other player moves. This reflects the assumption that a player who plans to wait until time t to move does not have to <u>commit</u> himself to moving until he has observed the history of the game up to time t. Consequently, if the other player moves before time t, the first player can react optimally, independently of what he had planned to do had the other player not moved at time t. A "silent" game of timing is one in which each player must commit himself to a time at which he will move independently of the action of the other at the outset of the game. Reinganum (1981a,b) has modeled the problem of when to adopt a new technology as a silent game of timing.

[2] This is just a normalization. A game with an infinite horizon can be converted into this framework by a simple change of variable such as $t = z/(1+z)$, where $z \in [0,\infty)$.

[3] It is not essential that $F_\alpha(1)$ and $L_\alpha(1)$ be defined since only return which can be realized at $t = 1$ is $S_\alpha(1)$. Defining $L_\alpha(t)$ and $F_\alpha(t)$ to be continuous at 1 merely allows us to identify $\lim_{t\uparrow 1}L_\alpha(t)$ with $L_\alpha(1)$ and $\lim_{t\uparrow 1}F_\alpha(t)$ with $F_\alpha(1)$.

[4] Generally, the game is a noisy duel, if, in addition to satisfying Assumption A1, F_α is assumed to be strictly decreasing and L_α strictly increasing. Since, in equilibrium, player α will never move at time t unless $F_\alpha(t) \le \max\{S_\alpha(t),L_\alpha(t)\}$, there is little loss in assuming that $F_\alpha(t) \le L_\alpha(t)$, given that we are assuming $S_\alpha(t) < L_\alpha(t)$ for $t \in [0,1]$.

[5] By a probability distribution on $[t,1]$, we mean any right-continuous nondecreasing function G from $(-\infty,\infty]$ to $[0,1]$ with $G(t) = 0$ for $t < 0$ and $G(1) = 1$. Throughout this paper, we will adopt the convention that

$$\int_v^t f(s)dG = \lim_{u\uparrow t}\int_v^u f(s)dG.$$

That is, integral does not include any mass points at times v or t.

[6] If $\int_{t_0}^{t_1}[dL_\alpha(v)/(F_\alpha(v)-L_\alpha(v))]dG_\alpha(v)$ does not exist, then define

$\int_{t_0}^{t_1}[dL_\alpha/(F_\alpha-L_\alpha]dG_\alpha = \lim_{t\downarrow t_0}\int_t^{t_1}[dL_\alpha/(F_\alpha-L_\alpha]dG_\alpha$. See footnote 5.

[7] $I_\beta(0,0)$ is either 0 or 1. Also, Assumption A2 implies that $I_\beta(0,0) = 1$ if $F_\alpha(0) \le S_\alpha(0)$.

[8] $I_\alpha(1,1)$ is either 0 or 1.

[9] It is not clear that their <u>strategies</u> correspond to any Nash equilibrium for the discrete time analogue with only a <u>finite</u> number of periods. However, one might interpret their equilibrium as a discrete time correlated equilibrium (Aumann (1974)) consisting of a mixture of two Nash equilibria in which one player moves immediately and the other player immediately after.

[10] Farrell and Saloner actually suppose that opportunities for adopting a new technology arrive to each firm according to independent Poisson processes. Consequently, only one player has the option to lead at the outset of the game but is uncertain about when the other player will have an option to lead. Under their assumptions, there is an equilibrium in which the firm adopts immediately.

[11] For example, if time is denoted by z, let $t = z/(z-1)$.

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